

Unitary Quantum Error Correction without Error Detection

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Abstract

We propose a quantum error correction without error detection. A quantum state ρ_0 combined with an ancilla state σ is encoded unitarily and an error operator is applied on the encoded state. The recovery operation then produces a tensor product state $\rho_0 \otimes \sigma'$. The decoding operation is combined with the recovery operation and the state ρ_0 is directly reproduced without referring to the code word. A higher rank projection operator required for a conventional operator quantum error correction is not necessary to implement. Encoding and the recovery operations are implemented with unitary operators only, which makes quantum error correction much easier than any other proposals.

A quantum system is vulnerable against disturbance from the environment. Environmental disturbance works as a source of decoherence and it must be suppressed to realize a working scalable quantum computer. Among many proposals to fight against decoherence, quantum error correcting code (QECC) is one of the most promising strategy to overcome decoherence. There are two main approaches to QECC to date. One employs syndrome measurements with extra ancilla qubits,[1, 2] while the other does not.[3] The latter is known as the operator quantum error correction (OQEC). In both approaches, a qubit state ρ_0 together with ancilla qubit state σ are encoded in the code space \mathcal{C} as $\rho = U_E(\rho_0 \otimes \sigma)$, on which an error operator \mathcal{E} acts subsequently. Here U_E is the encoding unitary operator. The recovery operator \mathcal{R} is applied on the state with an error to reproduce the initial code word ρ . Then a decoding operator U_E^{-1} is applied to obtain $\rho_0 \otimes \sigma$. In other words, QECC works if it satisfies

$$\mathcal{R}(\mathcal{E}(\rho)) = \rho \quad (\rho \in \mathcal{C}). \quad (1)$$

This is certainly a sufficient condition to reproduce the qubit state ρ_0 via recovering of the code word ρ .

It is the purpose of this Letter to propose a more efficient QECC. We replace the condition (1) by

$$\mathcal{R}(\mathcal{E}(U_E(\rho_0 \otimes \sigma))) = \rho_0 \otimes \sigma', \quad (2)$$

where σ' is an output ancilla state which depends on σ and the error operator \mathcal{E} . An essential observation is that the output state is a tensor product of ρ_0 and σ' . It is also important to note that although we use the code space in encoding a qubit state, it is not referred to in the recovery process. The recovery operation here involves the decoding process without going through the code space. Since the output state is a tensor product state, we can discard the ancillas without disturbing the qubit state ρ_0 .

Let us examine the simplest bit-flip error channel. A qubit state $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ is encoded by introducing two ancilla qubits as

$$U_E(|\psi_0\rangle|00\rangle) = \alpha|0\rangle_L + \beta|1\rangle_L \equiv |\psi\rangle, \quad (3)$$

where $|0\rangle_L = |000\rangle$ and $|1\rangle_L = |111\rangle$ are logical qubit basis vectors. We introduce the density matrices $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho = |\psi\rangle\langle\psi|$ to denote these pure states.

Now the error operator acts on ρ as

$$\Phi(\rho) = p_0\rho + \sum_{i=1}^3 p_i X_i \rho X_i \equiv \rho', \quad (4)$$

where $\sum_{i=0}^3 p_i = 1$ and X_i stands for the bit-flip operator acting on the i -th qubit. ($X_1 = X \otimes I_2 \otimes I_2$, for example, where $X = \sigma_x$ and I_n is the unit matrix of order n .) Here p_i is the probability with which X_i acts on ρ while p_0 is the probability with which ρ is left intact. We formally introduce $X_0 = I_2^{\otimes 3}$ to denote the latter process so that (4) is rewritten as

$$\Phi(\rho) = \sum_{i=0}^3 p_i X_i \rho X_i. \quad (5)$$

The conventional error recovery operation is studied in detail for this channel in [4], in which a superoperator Ψ is used as

$$\Psi(\rho') = P_{07} \left(\sum_{i=0}^3 X_i \rho' X_i \right) P_{07}, \quad (6)$$

where P_{ij} is a rank-2 projection operator onto a subspace $\text{Span}(|i\rangle, |j\rangle)$ with decimal indices i and j . Explicitly, it is given by

$$P_{07} = |0\rangle_{LL}\langle 0| + |1\rangle_{LL}\langle 1| = \text{diag}(1, 0, 0, 0, 0, 0, 0, 1).$$

The recovery operation introduced here is a superoperator, whose physical realization is challenging compared to unitary operations. Furthermore the rank-2 projection operator is also difficult for physical implementation in general.

These difficulties are avoided if the recovery operation is implemented with unitary matrices only and the unitary matrix acting on the error state ρ' outputs a tensor product state $\rho_0 \otimes \sigma'$. We simply discard σ' since it does not carry any useful information. Let us write

down the density matrix $\rho' = \mathcal{E}(\rho)$ explicitly to find the recovering unitary matrix;

$$\begin{aligned} \rho' &= \sum_{i=0}^3 p_i X_i \rho X_i \\ &= \begin{pmatrix} p_0|\alpha|^2 & 0 & 0 & 0 & 0 & 0 & 0 & p_0\alpha\beta^* \\ 0 & p_3|\alpha|^2 & 0 & 0 & 0 & 0 & p_3\alpha\beta^* & 0 \\ 0 & 0 & p_2|\alpha|^2 & 0 & 0 & p_2\alpha\beta^* & 0 & 0 \\ 0 & 0 & 0 & p_1|\beta|^2 & p_1\alpha^*\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1\alpha\beta^* & p_1|\alpha|^2 & 0 & 0 & 0 \\ 0 & 0 & p_2\alpha^*\beta & 0 & 0 & p_2|\beta|^2 & 0 & 0 \\ 0 & p_3\alpha^*\beta & 0 & 0 & 0 & 0 & p_3|\beta|^2 & 0 \\ p_0\alpha^*\beta & 0 & 0 & 0 & 0 & 0 & 0 & p_0|\beta|^2 \end{pmatrix} \end{aligned} \quad (7)$$

By inspecting the above matrix, we immediately notice that a permutation $P_{(3,4)}$ of two basis vectors $|011\rangle = |3\rangle$ and $|100\rangle = |4\rangle$, followed by a reverse ordering operation $P_{(4,5,6,7)}$ of basis vectors $|4\rangle, |5\rangle, |6\rangle$ and $|7\rangle$ maps ρ' a tensor product form;

$$\begin{aligned} &R \rho' R^\dagger \\ &= \begin{pmatrix} p_0|\alpha|^2 & 0 & 0 & 0 & p_0\alpha\beta^* & 0 & 0 & 0 \\ 0 & p_3|\alpha|^2 & 0 & 0 & 0 & p_3\alpha\beta^* & 0 & 0 \\ 0 & 0 & p_2|\alpha|^2 & 0 & 0 & 0 & p_2\alpha\beta^* & 0 \\ 0 & 0 & 0 & p_1|\alpha|^2 & 0 & 0 & 0 & p_1\alpha\beta^* \\ p_0\alpha^*\beta & 0 & 0 & 0 & p_0|\beta|^2 & 0 & 0 & 0 \\ 0 & p_3\alpha^*\beta & 0 & 0 & 0 & p_3|\beta|^2 & 0 & 0 \\ 0 & 0 & p_2\alpha^*\beta & 0 & 0 & 0 & p_2|\beta|^2 & 0 \\ 0 & 0 & 0 & p_1\alpha^*\beta & 0 & 0 & 0 & p_1|\beta|^2 \end{pmatrix} \\ &= \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \otimes \begin{pmatrix} p_0 & 0 & 0 & 0 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_1 \end{pmatrix}, \end{aligned} \quad (8)$$

where

$$R = P_{(4,5,6,7)} P_{(3,4)} = P_{(3,7)} P_{(4,5,6,7)}. \quad (9)$$

Although the second matrix in the bottom of Eq. (8) depends on the error operator, the first matrix reproduces the initial state exactly. Since the recovered state is a tensor product of

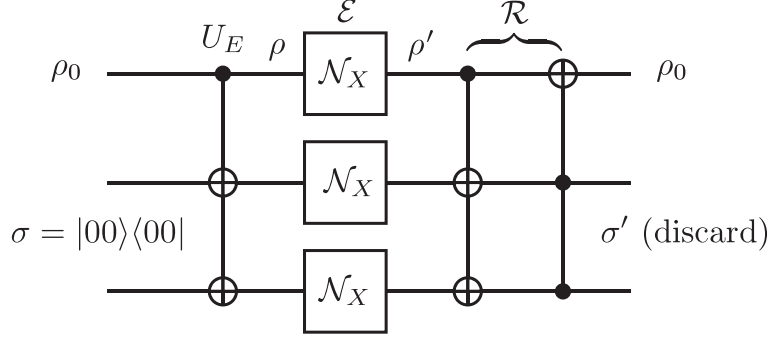


FIG. 1. 3-qubit QECC for a bit-flip channel. The input state is $\rho_0 \otimes \sigma$ while the output state is $\rho_0 \otimes \sigma'$. U_E, \mathcal{E} and \mathcal{R} are encoding circuit, error operation, and the ecovery operation, respectively. Needless to say, the unitary matrices act on both sides of a density matrix (adjoint representation). \mathcal{N}_X stands for the bit-flip noise.

ρ_0 and an ancillary state, we can safely discard the ancillas without leaving any trace on the first qubit and error correction is done. This last step corresponds to a rank-2 projection, whose physical realization is trivial in our scheme.

The permutation matrices $P_{(4,5,6,7)}$ and $P_{(3,7)}$ are nothing but $[C_1X_2X_3]$ and $[X_1C_2C_3]$ gates, respectively. Here we introduced the notation in which C_i means that the i th qubit works as a control bit in the gate. For example, $[C_1X_2X_3]$ stands for the controlled-NOT-NOT gate in conventional nomenclature, which has been used for encoding in Eq. (3). In summary, encoding, error and recovery operations for this QECC are depicted as Fig 1

It is instructive to rewrite the recovery operator in a different form to obtain a hint to find a recovery matrix R for more complicated cases. Let $\{|i\rangle\}$ be the set of basis vectors arranged as $\{|000\rangle, |001\rangle, |010\rangle, \dots, |110\rangle, |111\rangle\}$ and $\{|i'\rangle\}$ be the set of basis vector after the permutation operation $P_{(4,5,6,7)}P_{(3,4)}$ is applied, namely,

$$\begin{aligned} \{|i'\rangle\} &= \{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |111\rangle, |110\rangle, |101\rangle, |011\rangle\} \\ &= \{|0\rangle_L, X_3|0\rangle_L, X_2|0\rangle_L, X_1|0\rangle_L, |1\rangle_L, X_3|1\rangle_L, X_2|1\rangle_L, X_1|1\rangle_L\}. \end{aligned} \quad (10)$$

Then the recovery operator has the matrix elements $R_{ij} = \langle i'|j\rangle$. Explicitly R has the matrix

form

$$R = \begin{pmatrix} {}_L\langle 0| \\ {}_L\langle 0|X_3 \\ {}_L\langle 0|X_2 \\ {}_L\langle 0|X_1 \\ {}_L\langle 1| \\ {}_L\langle 1|X_3 \\ {}_L\langle 1|X_2 \\ {}_L\langle 1|X_1 \end{pmatrix}. \quad (11)$$

Note that this recovery matrix acts on a state vector with an error X_i as

$$RX_i|\psi\rangle = |\psi_0\rangle \otimes |**\rangle \quad (12)$$

due to the orthonormality ${}_L\langle m|X_i^T X_j|n\rangle_L = \delta_{ij}\delta_{mn}$. The ancilla state $|**\rangle$ depend on i , i.e., it tells us which error operator has acted on $|\psi\rangle$.

Now we are ready to extend our result to more complicated QECC, such as Shor's 9-qubit QECC[1] or the DiVincenzo-Shor 5-qubit QECC.[2] Let us work out the latter QECC for definiteness. Now the error operator \mathcal{E} involves X_i, Y_i and Z_i , where Y_i and Z_i stand for $Y = -i\sigma_y$ and $Z = \sigma_z$, respectively, acting on the i th qubit. Let U_E be the encoding operator

$$U_E[(\alpha|0\rangle + \beta|1\rangle)|0000\rangle] = \alpha|0\rangle_L + \beta|1\rangle_L, \quad (13)$$

where the logical qubit basis vectors are

$$\begin{aligned} |0\rangle_L = \frac{1}{4} [& |00000\rangle + |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10001\rangle \\ & - |10100\rangle - |01010\rangle - |00101\rangle - |10010\rangle - |01001\rangle \\ & - |11110\rangle - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle] \end{aligned} \quad (14)$$

and

$$\begin{aligned} |1\rangle_L = \frac{1}{4} [& |11111\rangle + |00111\rangle + |10011\rangle + |11001\rangle + |11100\rangle + |01110\rangle \\ & - |01011\rangle - |10101\rangle - |11010\rangle - |01101\rangle - |10110\rangle \\ & - |00001\rangle - |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle]. \end{aligned} \quad (15)$$

Let \mathcal{E} be the error operator which is expressed in terms of the operators

$$\{I_{32}, X_1, X_2, \dots, X_5, Y_1, Y_2, \dots, Y_5, Z_1, Z_2, \dots, Z_5\}. \quad (16)$$

Here $X_1 = X \otimes I^{\otimes 4}$ for example. It turns out to be convenient to rename the above operators as $\{W_i\}_{0 \leq i \leq 15}$. For example, we have $W_0 = I_{32}, W_1 = X_1, \dots, W_{15} = Z_5$. Suppose

$$\rho' = \mathcal{E}(\rho) = \sum_{i=0}^{15} p_i W_i \rho W_i^T \quad (17)$$

be the error state to be recovered. We try the following recovery matrix, which is inspired by the 3-qubit bit-flip QECC example,

$$R = \begin{pmatrix} |0\rangle_L^T \\ (W_1|0\rangle_L)^T \\ (W_2|0\rangle_L)^T \\ \vdots \\ (W_{15}|0\rangle_L)^T \\ |1\rangle_L^T \\ (W_1|1\rangle_L)^T \\ (W_2|1\rangle_L)^T \\ \vdots \\ (W_{15}|1\rangle_L)^T \end{pmatrix}. \quad (18)$$

The application of R on an error state $\mathcal{E}(\rho)$ outputs the following state

$$R\rho'R^T = \rho_0 \otimes \sigma', \quad (19)$$

where $\rho_0 = |\psi_0\rangle\langle\psi_0|$ as before and

$$\sigma' = \text{diag}(p_0, p_1, \dots, p_{14}, p_{15}). \quad (20)$$

A quantum circuit which implements the five-qubit QECC is shown in Fig. 2, in which the encoding circuit is taken from [5].

The matrix R is orthogonal and can be implemented with elementary quantum gates such as one-qubit gates and CNOT gates in principle. Nonetheless, it is not a simple permutation gate any more due to the complicated structure of the logical qubit states $|0\rangle_L$ and $|1\rangle_L$ and its implementation must be challenging. Let us look at Fig. 1 to find a hint to overcome this problem. The recovery operation is made of the inverse encoding circuit $[C_1 X_1 X_2]$ and a permutation of basis vectors given by the controlled-controlled NOT gate $[X_1 C_2 C_3]$. We also tried the inverse encoding circuit U_E^{-1} in our 5-qubit. Then it turned out that the first

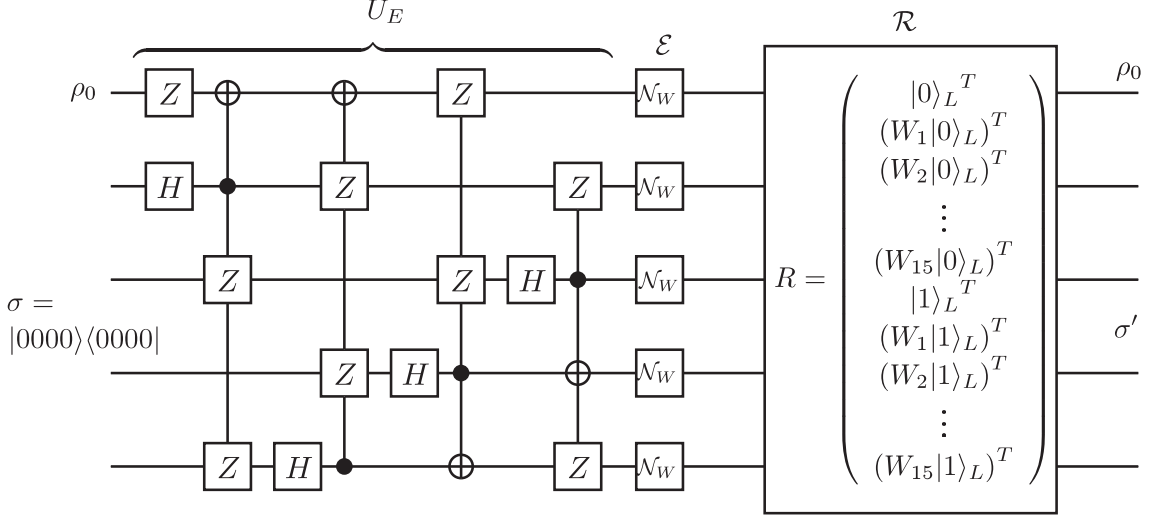


FIG. 2. 5-qubit QECC. The input state is $\rho_0 \otimes |0000\rangle\langle 0000|$ while the output state is $\rho_0 \otimes \sigma'$. U_E , \mathcal{E} and \mathcal{R} are encoding circuit, error operation, and recovery operation, respectively. \mathcal{N}_W stands for the 15 noise operators.

qubit state does not agree with $|\psi_0\rangle$ exactly for some errors and we need additional bit-flips and/or phase-flips to correct this. Circuit implementation of this recovery operation will be reported elsewhere.

Shor's 9-qubit QECC is also implemented with a unitary recovery matrix. It is a trivial generalization of the 3-qubit bit-flip QECC and we simply give the circuit for this case in Fig. 3 without giving the lengthy details, which will be reported elsewhere.

In summary, we proposed an efficient implementation of QECC, whose recovery process involves unitary operations only. No syndrome readouts nor higher-rank projection operators are required. This makes physical implementation of QECC considerably easier. We have demonstrated our proposal with 3-qubit bit-flip QECC, DiVincenzo-Shor's 5-qubit QECC and Shor's 9-qubit QECC. Details of our QECC, involving efficient decomposition of the recovery operation into elementary gates, are in preparation and will be reported elsewhere.

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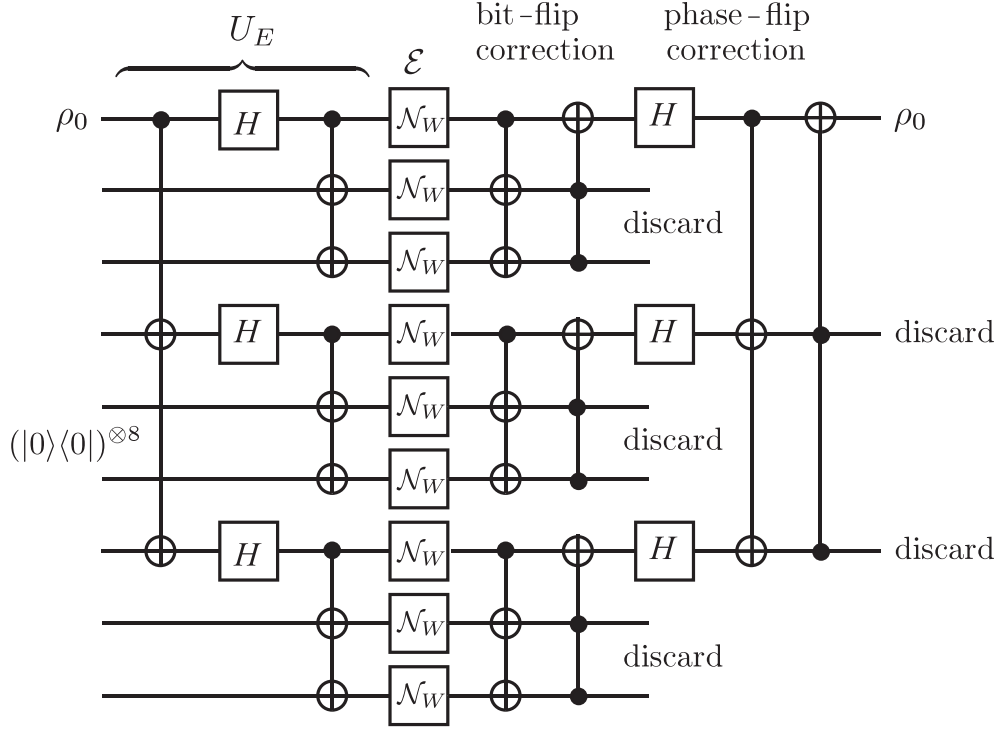


FIG. 3.

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